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EXPONENTIAL SUMS ON $\mathbf{A}^n$, II

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Abstract. We prove a vanishing theorem for the p-adic cohomology of exponential sums on $\mathbf{A}^n$. In particular, we obtain new classes of exponential sums on $\mathbf{A}^n$ that have a single nonvanishing p-adic cohomology group. The dimension of this cohomology group equals a sum of Milnor numbers.

1. Introduction

Let p be a prime number, $q = p^a$, and let $\mathbf{F}_q$ be the finite field of q elements. Associated to a polynomial $f \in \mathbf{F}_q[x_1, \dots, x_n]$ and a nontrivial additive character $\Psi : \mathbf{F}_q \to \mathbf{C}^\times$ are exponential sums

$$S(\mathbf{A}^n(\mathbf{F}_{q^i}), f) = \sum_{x_1,\dots,x_n \in \mathbf{F}_{q^i}} \Psi(\mathrm{Trace}_{\mathbf{F}_{q^i}/\mathbf{F}_q} f(x_1, \dots, x_n)) \tag{1.1}$$

and an L-function

$$L(\mathbf{A}^n, f; t) = \exp\left(\sum_{i=1}^{\infty} S(\mathbf{A}^n(\mathbf{F}_{q^i}), f)\frac{t^i}{i}\right). \tag{1.2}$$

One of the basic results on exponential sums is the following theorem of Deligne [3, Théorème 8.4]. Let $\delta = \deg f$ and write

$$f = f^{(\delta)} + f^{(\delta-1)} + \cdots + f^{(0)}, \tag{1.3}$$

where $f^{(j)}$ is homogeneous of degree j.

Theorem 1.4. *Suppose $(p, \delta) = 1$ and $f^{(\delta)} = 0$ defines a smooth hypersurface in $\mathbf{P}^{n-1}$. Then $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ is a polynomial of degree $(\delta - 1)^n$, all of whose reciprocal roots have absolute value $q^{n/2}$.*

For exponential sums on $\mathbf{A}^n$, several generalizations of Deligne's result have been proved ([1], [2], [5], [7]). In all these theorems, the hypothesis implies that f, regarded as a function from $\mathbf{A}^n$ to $\mathbf{A}^1$, has only finitely many critical points and the degree of the polynomial $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ equals the sum of the Milnor numbers of those critical points. We examine these critical points more closely.

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We write $\mathbf{F}_q[x]$ for $\mathbf{F}_q[x_1,\dots,x_n]$ and consider the complex $(\Omega^{\cdot}_{\mathbf{F}_q[x]/\mathbf{F}_q},\phi_f)$, where $\Omega^k_{\mathbf{F}_q[x]/\mathbf{F}_q}$ denotes the module of differential k-forms of $\mathbf{F}_q[x_1,\dots,x_n]$ over $\mathbf{F}_q$ and $\phi_f:\Omega^k_{\mathbf{F}_q[x]/\mathbf{F}_q}\to\Omega^{k+1}_{\mathbf{F}_q[x]/\mathbf{F}_q}$ is defined by

$$\phi_f(\omega)=df\wedge\omega,$$

where $d:\Omega^k_{\mathbf{F}_q[x]/\mathbf{F}_q}\to\Omega^{k+1}_{\mathbf{F}_q[x]/\mathbf{F}_q}$ is the exterior derivative. The map $f:\mathbf{A}^n\to\mathbf{A}^1$ has only isolated critical points if and only if

$$H^i(\Omega^{\cdot}_{\mathbf{F}_q[x]/\mathbf{F}_q},\phi_f)=0\qquad\text{for } i\neq n, \tag{1.5}$$

which implies that $\dim_{\mathbf{F}_q}H^n(\Omega^{\cdot}_{\mathbf{F}_q[x]/\mathbf{F}_q},\phi_f)$ is finite. Since

$$H^n(\Omega^{\cdot}_{\mathbf{F}_q[x]/\mathbf{F}_q},\phi_f)\simeq\mathbf{F}_q[x_1,\dots,x_n]/(\partial f/\partial x_1,\dots,\partial f/\partial x_n),$$

we have

$$\dim_{\mathbf{F}_q}H^n(\Omega^{\cdot}_{\mathbf{F}_q[x]/\mathbf{F}_q},\phi_f)=M_f,$$

where M_f denotes the sum of the Milnor numbers of the critical points of f.

We consider one approach to verifying (1.5). Every $\omega\in\Omega^k_{\mathbf{F}_q[x]/\mathbf{F}_q}$ can be uniquely written in the form

$$\omega=\sum_{1\le i_1<\cdots<i_k\le n}\omega(i_1,\dots,i_k)\,dx_{i_1}\wedge\cdots\wedge dx_{i_k},$$

with $\omega(i_1,\dots,i_k)\in\mathbf{F}_q[x]$. Define

$$\deg_{\text{coeff}}\omega=\max_{1\le i_1<\cdots<i_k\le n}\{\deg\omega(i_1,\dots,i_k)\},$$
$$\deg\omega=\deg_{\text{coeff}}\omega+(n-k)(\delta-1).$$

The point of the latter definition is that we can define an increasing filtration $F.$ on $\Omega^k_{\mathbf{F}_q[x]/\mathbf{F}_q}$ by setting

$$F_l\Omega^k_{\mathbf{F}_q[x]/\mathbf{F}_q}=\{\omega\in\Omega^k_{\mathbf{F}_q[x]/\mathbf{F}_q}\mid\deg\omega\le l\}$$

and $(\Omega^{\cdot}_{\mathbf{F}_q[x]/\mathbf{F}_q},\phi_f)$ then becomes a filtered complex. Consider the associated spectral sequence

$$E_1^{r,s}=H^{r+s}(F_r/F_{r-1}(\Omega^{\cdot}_{\mathbf{F}_q[x]/\mathbf{F}_q},\phi_f))\Rightarrow H^{r+s}(\Omega^{\cdot}_{\mathbf{F}_q[x]/\mathbf{F}_q},\phi_f). \tag{1.6}$$

The E_1-terms are just the cohomology of the homogeneous pieces of the associated graded complex to $(\Omega^{\cdot}_{\mathbf{F}_q[x]/\mathbf{F}_q},\phi_f)$, which may be identified with $(\Omega^{\cdot}_{\mathbf{F}_q[x]/\mathbf{F}_q},\phi_{f^{(\delta)}})$. Since this latter complex is isomorphic to the Koszul complex on $\mathbf{F}_q[x_1,\dots,x_n]$ defined by $\{\partial f^{(\delta)}/\partial x_i\}_{i=1}^n$, the assertion that

$$E_1^{r,s}=0\qquad\text{for } r+s\neq n \tag{1.7}$$

is equivalent to the assertion that

$$\{\partial f^{(\delta)}/\partial x_i\}_{i=1}^n\quad\text{form a regular sequence in }\mathbf{F}_q[x_1,\dots,x_n]. \tag{1.8}$$

It follows from (1.8) that

$$\dim_{\mathbf{F}_q}\mathbf{F}_q[x_1,\dots,x_n]/(\partial f^{(\delta)}/\partial x_1,\dots,\partial f^{(\delta)}/\partial x_n)=(\delta-1)^n;$$

hence

$$\dim_{\mathbf{F}_q}\bigoplus_{r+s=n}E_1^{r,s}=(\delta-1)^n. \tag{1.9}$$

The spectral sequence (1.6) and condition (1.7) imply that (1.5) holds, and (1.9) then implies that $M_f = (\delta - 1)^n$. Theorem 1.4 of [2], a slight generalization of Theorem 1.4 above, can then be reformulated as follows.

Theorem 1.10. *Suppose* (1.7) *holds. Then* $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ *is a polynomial of degree* M_f*, all of whose reciprocal roots have absolute value* $q^{n/2}$.

Condition (1.5) holds if there exists a positive integer e such that

$$E_e^{r,s} = 0 \qquad \text{for } r + s \neq n. \tag{1.11}$$

We are interested in determining the extent to which the conclusion of Theorem 1.10 holds when condition (1.7) is replaced by condition (1.11) for some $e > 1$. In general, some additional hypothesis is needed, as is illustrated by the one-variable example $f(x_1) = x_1^p - x_1$ over the field $\mathbf{F}_p$. The purpose of this paper is to prove a result that provides evidence for such a theorem.

Dwork has associated to f a complex $(\Omega^{\cdot}_{C(b)}, D)$ of length n depending on a choice of rational parameter b satisfying $0 < b < p/(p-1)$ (we review this theory in section 2). Each $\Omega^i_{C(b)}$, $i = 0, \ldots, n$, is a p-adic Banach space over a field $\tilde{\Omega}_0$ (a finite extension of $\mathbf{Q}_p$) and is equipped with a Frobenius operator α_i commuting with the differential D of the complex. Furthermore,

$$L(\mathbf{A}^n, f; t) = \prod_{i=0}^{n} \det(I - t\alpha_i \mid H^i(\Omega^{\cdot}_{C(b)}, D))^{(-1)^{i+1}}. \tag{1.12}$$

Theorem 1.13. *Suppose there exist* e, m *such that* $E_e^{r,s} = 0$ *for all* r, s *satisfying* $r + s = m$. *Then for*

$$\frac{\delta}{(p-1)(\delta - e + 1)} < b < \frac{p\delta}{(p-1)\delta + e - 1} \tag{1.14}$$

we have

$$H^m(\Omega^{\cdot}_{C(b)}, D) = 0.$$

If (1.11) *holds, then, in addition, for* b *in the range* (1.14) *we have*

$$\dim_{\tilde{\Omega}_0} H^n(\Omega^{\cdot}_{C(b)}, D) = M_f.$$

Remark. It is easily seen that in (1.14) the upper bound for b is greater than the lower bound for b if and only if

$$\left(1 + \frac{p}{(p-1)^2}\right)(e-1) < \delta, \tag{1.15}$$

i.e., (1.15) is equivalent to the existence of a rational b satisfying (1.14). For example, if $e = 2$, then this condition requires $\delta \geq 2$ for odd primes p and $\delta \geq 4$ for $p = 2$. In general, for p sufficiently large relative to δ, it becomes simply $e \leq \delta$.

The finite-dimensionality of $H^n(\Omega^{\cdot}_{C(b)}, D)$ implies, by [11, section 3.4], that α_n is invertible on $H^n(\Omega^{\cdot}_{C(b)}, D)$. So Theorem 1.13 and equation (1.12) give the following.

Corollary 1.16. *Suppose* (1.11) *holds for a positive integer* e *satisfying* (1.15). *Then* $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ *is a polynomial of degree* M_f.

The main idea in the proof of Theorem 1.13 is to relate the spectral sequence (1.6) to the spectral sequence associated to the filtration by p-divisibility on the complex $(\Omega^{\cdot}_{C(b)}, D)$. This is accomplished by Theorem 3.6, which is applied in section 4 to compute the cohomology of $(\Omega^{\cdot}_{C(b)}, D)$.

We conjecture that the hypothesis of Corollary 1.16 implies that all reciprocal roots of $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ have absolute value $q^{n/2}$. This would imply, in particular, the estimate

$$|S(\mathbf{A}^n(\mathbf{F}_{q^i}), f)| \leq M_f q^{ni/2}.$$

Typically, such results are proved by computing the corresponding l-adic cohomology groups. However, we have been unable to apply our previous method [1], [2] for calculating l-adic cohomology from p-adic cohomology because we have been unable to compute the p-divisibility of the Frobenius determinant under the hypothesis of Corollary 1.16.

It is an interesting problem to find geometric conditions that imply (1.11) for some $e > 1$, and we plan to return to this question in a future article. As an example, we prove in section 5 the following. Make (1.3) more precise by writing

$$f = f^{(\delta)} + f^{(\delta')} + f^{(\delta'-1)} + \cdots + f^{(0)}, \tag{1.17}$$

where $f^{(j)}$ is homogeneous of degree j and $1 \leq \delta' \leq \delta - 1$, i.e., $f^{(\delta')}$ is the homogeneous part of second-highest degree of f.

Theorem 1.18. *Suppose that $f^{(\delta)} = f_1^{a_1} \cdots f_r^{a_r}$, where for every subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$ the system of equations*

$$f_{i_1} = \cdots = f_{i_k} = 0$$

defines a smooth complete intersection of codimension k in $\mathbf{P}^{n-1}$ and, if either $k \geq 2$ or $k = 1$ and $a_{i_1} > 1$, the system of equations

$$f^{(\delta')} = f_{i_1} = \cdots = f_{i_k} = 0$$

defines a smooth complete intersection of codimension $k+1$ in $\mathbf{P}^{n-1}$. Suppose also that $(p, \delta\delta' a_1 \cdots a_r) = 1$. Then (1.11) holds for $e = \delta - \delta' + 1$.

Our initial work on this topic was prompted by an l-adic result of García. It led to the following example, whose proof will appear elsewhere. We refer to [7] for the definitions of "weighted homogeneous" isolated singularity and "total degree" of a weighted homogeneous isolated singularity.

Theorem 1.19. *Suppose that the hypersurface $f^{(\delta)} = 0$ in $\mathbf{P}^{n-1}$ has at worst weighted homogeneous isolated singularities, of total degrees $\delta_1, \ldots, \delta_s$, and that none of these singularities lies on the hypersurface $f^{(\delta')} = 0$ in $\mathbf{P}^{n-1}$. Suppose also that $(p, \delta\delta'\delta_1 \cdots \delta_s) = 1$. Then (1.11) holds for $e = \delta - \delta' + 1$.*

The hypothesis of Theorem 1.19 (for $\delta' = \delta - 1$) first appears in [7]. García shows that it implies that the l-adic cohomology groups associated to the exponential sum (1.1) vanish except in degree n, where the cohomology group is pure of weight n and has dimension M_f. Thus the conclusion of Theorem 1.10 holds in this case. García's results and Theorem 1.19 are what originally led us to suspect that a result such as Theorem 1.13 should hold.

2. p-ADIC COHOMOLOGY

In this section we review the basic properties of Dwork's p-adic cohomology theory that we shall need. For a more detailed exposition of this material, we refer the reader to [2].

Let $\mathbf{Q}_p$ be the field of p-adic numbers, ζ_p a primitive p-th root of unity, and $\Omega_1 = \mathbf{Q}_p(\zeta_p)$. The field Ω_1 is a totally ramified extension of $\mathbf{Q}_p$ of degree $p-1$. Let K be the unramified extension of $\mathbf{Q}_p$ of degree a. Set $\Omega_0 = K(\zeta_p)$. The Frobenius automorphism $x \mapsto x^p$ of $\mathrm{Gal}(\mathbf{F}_q/\mathbf{F}_p)$ lifts to a generator τ of $\mathrm{Gal}(\Omega_0/\Omega_1)$ ($\simeq \mathrm{Gal}(K/\mathbf{Q}_p)$) by requiring $\tau(\zeta_p) = \zeta_p$. Let Ω be the completion of an algebraic closure of Ω_0. Denote by "ord" the additive valuation on Ω normalized by $\mathrm{ord}\, p = 1$ and by "ord_q" the additive valuation normalized by $\mathrm{ord}_q\, q = 1$.

Let $E(t)$ be the Artin–Hasse exponential series:

$$E(t) = \exp\biggl(\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i}\biggr).$$

Let $\gamma \in \Omega_1$ be a solution of $\sum_{i=0}^{\infty} t^{p^i}/p^i = 0$ satisfying $\mathrm{ord}\,\gamma = 1/(p-1)$ and put

$$\theta(t) = E(\gamma t) = \sum_{i=0}^{\infty} \lambda_i t^i \in \Omega_1[[t]]. \tag{2.1}$$

The series $\theta(t)$ is a splitting function [6, section 4a] whose coefficients satisfy

$$\mathrm{ord}\,\lambda_i \geq i/(p-1). \tag{2.2}$$

We consider the following spaces of p-adic functions. Let b be a positive rational number and choose a positive integer M such that Mb/p and $M\delta/(p(p-1))$ are integers. Let π be such that

$$\pi^{M\delta} = p \tag{2.3}$$

and put $\tilde{\Omega}_1 = \Omega_1(\pi)$, $\tilde{\Omega}_0 = \Omega_0(\pi)$. The element π is a uniformizing parameter for the rings of integers of $\tilde{\Omega}_1$ and $\tilde{\Omega}_0$. We extend $\tau \in \mathrm{Gal}(\Omega_0/\Omega_1)$ to a generator of $\mathrm{Gal}(\tilde{\Omega}_0/\tilde{\Omega}_1)$ by requiring $\tau(\pi) = \pi$. For $u = (u_1, \ldots, u_n) \in \mathbf{R}^n$, we put $|u| = u_1 + \cdots + u_n$. Define

$$C(b) = \biggl\{ \sum_{u \in \mathbf{N}^n} A_u \pi^{Mb|u|} x^u \mid A_u \in \tilde{\Omega}_0 \text{ and } A_u \to 0 \text{ as } u \to \infty \biggr\}. \tag{2.4}$$

For $\xi = \sum_{u\in\mathbf{N}^n} A_u \pi^{Mb|u|} x^u \in C(b)$, define

$$\mathrm{ord}\,\xi = \min_{u\in\mathbf{N}^n} \{\mathrm{ord}\, A_u\}.$$

Given $c \in \mathbf{R}$, we put

$$C(b,c) = \{\xi \in C(b) \mid \mathrm{ord}\,\xi \geq c\}.$$

Clearly, $C(b) = \bigcup_{c\in\mathbf{R}} C(b,c)$.

Let $\hat{f} = \sum_u \hat{a}_u x^u \in K[x_1, \ldots, x_n]$ be the Teichmüller lifting of the polynomial $f \in \mathbf{F}_q[x_1, \ldots, x_n]$, i.e., $(\hat{a}_u)^q = \hat{a}_u$ and the reduction of $\hat{f}$ modulo p is f. Set

$$F(x) = \prod_u \theta(\hat{a}_u x^u), \tag{2.5}$$

$$F_0(x) = \prod_{i=0}^{a-1} \prod_u \theta((\hat{a}_u x^u)^{p^i}). \tag{2.6}$$

The estimate (2.2) implies that $F \in C(b,0)$ for all $b < 1/(p-1)$ and $F_0 \in C(b,0)$ for all $b < p/(q(p-1))$. Define an operator ψ on formal power series by

$$\psi\left(\sum_{u\in\mathbf{N}^n} A_u x^u\right) = \sum_{u\in\mathbf{N}^n} A_{pu} x^u. \tag{2.7}$$

It is clear that $\psi(C(b,c)) \subseteq C(pb,c)$. For $0 < b < p/(p-1)$, let $\alpha = \psi^a \circ F_0$ be the composition

$$C(b) \hookrightarrow C(b/q) \xrightarrow{F_0} C(b/q) \xrightarrow{\psi^a} C(b).$$

Then α is a completely continuous $\tilde{\Omega}_0$-linear endomorphism of $C(b)$. We shall also need to consider $\beta = \tau^{-1} \circ \psi \circ F$, which is a completely continuous $\tilde{\Omega}_1$-linear (or $\tilde{\Omega}_0$-semilinear) endomorphism of $C(b)$. Note that $\alpha = \beta^a$.

Set $\hat{f}_i = \partial\hat{f}/\partial x_i$ and let $\gamma_l = \sum_{i=0}^{l} \gamma^{p^i}/p^i$. By the definition of γ, we have

$$\operatorname{ord} \gamma_l \geq \frac{p^{l+1}}{p-1} - l - 1. \tag{2.8}$$

For $i = 1, \ldots, n$, define differential operators D_i by

$$D_i = \pi^{Mb(\delta-1)}\gamma^{-1}\left(\frac{\partial}{\partial x_i} + H_i\right), \tag{2.9}$$

where

$$H_i = \sum_{l=0}^{\infty} \gamma_l p^l x_i^{p^l-1} \hat{f}_i^{\tau^l}(x^{p^l}) \in C\left(b, \frac{1}{p-1} - b\frac{\delta-1}{\delta}\right) \tag{2.10}$$

for $b < p/(p-1)$. Thus D_i and "multiplication by H_i" operate on $C(b)$ for $b < p/(p-1)$. As explained in [2], we have

$$\alpha \circ x_i D_i = q x_i D_i \circ \alpha, \tag{2.11}$$

$$\beta \circ x_i D_i = p x_i D_i \circ \beta. \tag{2.12}$$

The significance of the normalizing factor $\pi^{Mb(\delta-1)}\gamma^{-1}$ will be explained below.

Consider the de Rham-type complex $(\Omega^{\cdot}_{C(b)}, D)$, where

$$\Omega^k_{C(b)} = \bigoplus_{1\leq i_1<\cdots<i_k\leq n} C(b)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and $D : \Omega^k_{C(b)} \to \Omega^{k+1}_{C(b)}$ is defined by

$$D(\xi\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \left(\sum_{i=1}^{n} D_i(\xi)\, dx_i\right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

We extend the mapping α to a mapping $\alpha_\cdot : \Omega^{\cdot}_{C(b)} \to \Omega^{\cdot}_{C(b)}$ defined by linearity and the formula

$$\alpha_k(\xi\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = q^{n-k}\frac{1}{x_{i_1}\cdots x_{i_k}}\alpha(x_{i_1}\cdots x_{i_k}\xi)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Equation (2.11) implies that $\alpha_\cdot$ is a map of complexes. The Dwork trace formula, as formulated by Robba [11], then gives

$$L(\mathbf{A}^n, f; t) = \prod_{k=0}^{n} \det(I - t\alpha_k \mid \Omega^k_{C(b)})^{(-1)^{k+1}}. \tag{2.13}$$

This implies (using [13, Proposition 9])

$$L(\mathbf{A}^n, f; t) = \prod_{k=0}^{n} \det(I - t\alpha_k \mid H^k(\Omega^{\cdot}_{C(b)}, D))^{(-1)^{k+1}}, \tag{2.14}$$

where we denote the induced map on cohomology by α_k also.

The p-adic Banach space $C(b)$ has a decreasing filtration $\{\hat{F}^r C(b)\}_{r=-\infty}^{\infty}$ of $\mathcal{O}_{\tilde{\Omega}_0}$-modules defined by setting

$$\hat{F}^r C(b) = \left\{ \sum_{u \in \mathbf{N}^n} A_u \pi^{Mb|u|} x^u \in C(b) \mid A_u \in \pi^r \mathcal{O}_{\tilde{\Omega}_0} \text{ for all } u \right\},$$

where $\mathcal{O}_{\tilde{\Omega}_0}$ denotes the ring of integers of $\tilde{\Omega}_0$. (In our earlier notation, $\hat{F}^r C(b) = C(b, r/M\delta)$.) We extend this to a filtration on $\Omega^{\cdot}_{C(b)}$ by defining

$$\hat{F}^r \Omega^k_{C(b)} = \bigoplus_{1 \le i_1 < \cdots < i_k \le n} \hat{F}^r C(b)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

This filtration is exhaustive and separated, i.e.,

$$\bigcup_{r \in \mathbf{Z}} \hat{F}^r \Omega^{\cdot}_{C(b)} = \Omega^{\cdot}_{C(b)} \quad \text{and} \quad \bigcap_{r \in \mathbf{Z}} \hat{F}^r \Omega^{\cdot}_{C(b)} = (0).$$

When $b \ge 1/(p-1)$, our choice of the normalizing factor $\pi^{Mb(\delta-1)}\gamma^{-1}$ in (2.9) guarantees that the D_i respect this filtration, i.e., $D_i(\hat{F}^r C(b)) \subseteq \hat{F}^r C(b)$; hence $D(\hat{F}^r \Omega^k_{C(b)}) \subseteq \hat{F}^r \Omega^{k+1}_{C(b)}$. Associated to the filtered complex $(\Omega^{\cdot}_{C(b)}, D)$ is the spectral sequence (for $b \ge 1/(p-1)$)

$$\hat{E}_1^{r,s} = H^{r+s}(\hat{F}^r \Omega^{\cdot}_{C(b)} / \hat{F}^{r+1} \Omega^{\cdot}_{C(b)}) \Rightarrow H^{r+s}(\Omega^{\cdot}_{C(b)}, D). \tag{2.15}$$

The notation $\hat{E}_t^{r,s}$ does not express the dependence of this spectral sequence on the choice of b; however, this should not cause confusion. We shall prove Theorem 1.13 by analyzing this spectral sequence.

For notational convenience we define an "exterior derivative" $d : \Omega^k_{C(b)} \to \Omega^{k+1}_{C(b)}$. It is characterised by $\tilde{\Omega}_0$-linearity and the formula

$$d(\xi\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \left(\sum_{i=1}^{n} \frac{\partial \xi}{\partial x_i}\, dx_i \right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Although we use the same symbol "d" for the exterior derivative on both $\Omega^{\cdot}_{C(b)}$ and $\Omega^{\cdot}_{\mathbf{F}_q[x]/\mathbf{F}_q}$, its meaning will be clear from the context.

3. Relation between the spectral sequences $E_t^{r,s}$ and $\hat{E}_t^{r,s}$

We begin with some general remarks on the filtration $\hat{F}^{\cdot}$ and the associated spectral sequence $\hat{E}_t^{r,s}$. Consider the operator D_i defined in (2.9). The terms of degree $< \delta - e$ in $\pi^{Mb(\delta-1)}\gamma^{-1}H_i$ lie in $\hat{F}^{Mbe}C(b)$. The upper bound on b given by (1.14) guarantees that the terms of degree $> \delta - 1$ in $\pi^{Mb(\delta-1)}\gamma^{-1}H_i$ lie in $\hat{F}^{Mb(e-1)+1}C(b)$. The lower bound on b given by (1.14) guarantees that

$\pi^{Mb(\delta-1)}\gamma^{-1}\partial/\partial x_i$ maps $\hat{F}^0 C(b)$ into $\hat{F}^{Mb(e-1)+1}C(b)$. Examining the terms of degrees $\delta-e,\dots,\delta-1$ in $\pi^{Mb(\delta-1)}\gamma^{-1}H_i$ then gives for $\xi\in\hat{F}^0C(b)$,

$$D_i(\xi)-\pi^{Mb(\delta-1)}\left(\sum_{j=0}^{e-1}\frac{\partial \hat{f}^{(\delta-j)}}{\partial x_i}\right)\xi\in\hat{F}^{Mb(e-1)+1}C(b). \tag{3.1}$$

It follows that if $\omega\in\hat{F}^r\Omega^k_{C(b)}$, $0\le k\le n$, then

$$D(\omega)-\pi^{Mb(\delta-1)}\left(\sum_{j=0}^{e-1}d\hat{f}^{(\delta-j)}\right)\wedge\omega\in\hat{F}^{r+Mb(e-1)+1}\Omega^{k+1}_{C(b)}. \tag{3.2}$$

Note that

$$\pi^{Mb(\delta-j-1)}d\hat{f}^{(\delta-j)}\in\hat{F}^0\Omega^1_{C(b)}\qquad\text{for } j=0,1,\dots,e-1. \tag{3.3}$$

We recall the definition of $\hat{E}^{r,s}_t$. Put

$$\hat{Z}^{r,s}_t=\{\omega\in\hat{F}^r\Omega^{r+s}_{C(b)}\mid D(\omega)\in\hat{F}^{r+t}\Omega^{r+s+1}_{C(b)}\}.$$

Then

$$\hat{E}^{r,s}_t=\frac{\hat{Z}^{r,s}_t+\hat{F}^{r+1}\Omega^{r+s}_{C(b)}}{D(\hat{Z}^{r-t+1,s+t-2}_{t-1})+\hat{F}^{r+1}\Omega^{r+s}_{C(b)}}.$$

A priori, our filtration $\hat{F}^{\cdot}$ on $\Omega^{\cdot}_{C(b)}$ is a filtration by subcomplexes of $\mathcal{O}_{\tilde{\Omega}_0}$-modules; hence the $\hat{E}^{s,t}_r$ are $\mathcal{O}_{\tilde{\Omega}_0}$-modules. However, the preceding equation shows that $\pi\hat{E}^{s,t}_r=0$; hence the $\hat{E}^{s,t}_r$ are naturally vector spaces over $\mathbf{F}_q=\mathcal{O}_{\tilde{\Omega}_0}/\pi\mathcal{O}_{\tilde{\Omega}_0}$.

Let $\phi\in\Omega^m_{\mathbf{F}_q[x]/\mathbf{F}_q}$, $\deg\phi=r$. By our definition of the degree of an m-form, we have

$$\phi=\sum_{1\le i_1<\cdots<i_m\le n}\left(\sum_{j=0}^{l}\sum_{|u|=j}\phi_u(i_1,\dots,i_m)x^u\right)dx_{i_1}\wedge\cdots\wedge dx_{i_m}$$

with $\phi_u(i_1,\dots,i_m)\in\mathbf{F}_q$, where $l=r-(n-m)(\delta-1)$. The condition $\phi\in Z^{r,m-r}_e$ means that for $i=0,1,\dots,e-1$,

$$\sum_{j=0}^{i}df^{(\delta-j)}\wedge\left(\sum_{1\le i_1<\cdots<i_m\le n}\ \sum_{|u|=l-i+j}\phi_u(i_1,\dots,i_m)x^u\,dx_{i_1}\wedge\cdots\wedge dx_{i_m}\right)=0. \tag{3.4}$$

For such a ϕ, we define its *normalized Teichmüller lifting* $\hat{\phi}\in\hat{F}^0\Omega^m_{C(b)}$ by

$$\hat{\phi}=\pi^{Mbl}\sum_{1\le i_1<\cdots<i_m\le n}\left(\sum_{j=0}^{l}\sum_{|u|=j}\hat{\phi}_u(i_1,\dots,i_m)x^u\right)dx_{i_1}\wedge\cdots\wedge dx_{i_m},$$

where $\hat{\phi}_u(i_1,\dots,i_m)\in K$ is the Teichmüller lifting of $\phi_u(i_1,\dots,i_m)\in\mathbf{F}_q$. We shall also have occasion to refer to the *nonnormalized Teichmüller lifting*, by which we mean the same expression but with the normalizing factor π^{Mbl} omitted.

To check that $\hat{\phi}\in\hat{Z}^{0,m}_{Mb(e-1)+1}$ (and hence $\pi^r\hat{\phi}\in\hat{Z}^{r,m-r}_{Mb(e-1)+1}$ for all $r\in\mathbf{Z}$, since $\pi^r\hat{F}^0=\hat{F}^r$), it suffices by (3.2) to show that

$$\pi^{Mb(\delta-1)}\sum_{j=0}^{e-1}d\hat{f}^{(\delta-j)}\wedge\hat{\phi}\in\hat{F}^{Mb(e-1)+1}\Omega^{m+1}_{C(b)}. \tag{3.5}$$

From the definition of $\hat{\phi}$, it follows that each term $x^v\, dx_{i_1} \wedge \cdots \wedge dx_{i_{m+1}}$ appearing in (3.5) has coefficient in $\pi^{Mb(\delta-1+l)}\mathcal{O}_K$. So for $|v| < \delta + l - e$, all these terms lie in $\hat{F}^{Mb(e-1)+1}\Omega^{m+1}_{C(b)}$. For $i = 0, 1, \ldots, e-1$, the terms with $|v| = l + \delta - 1 - i$ are given by

$$\pi^{Mb(l+\delta-1)} \sum_{j=0}^{i} d\hat{f}^{(\delta-j)} \wedge \left(\sum_{1 \le i_1 < \cdots < i_m \le n} \sum_{|u|=l-i+j} \hat{\phi}_u(i_1, \ldots, i_m) x^u\, dx_{i_1} \wedge \cdots \wedge dx_{i_m} \right).$$

It follows from (3.4) that each term $x^v\, dx_{i_1} \wedge \cdots \wedge dx_{i_{m+1}}$ appearing in this expression has coefficient in $\pi^{Mb(l+\delta-1)}p\mathcal{O}_K$ (K is an unramified extension of $\mathbf{Q}_p$). Equation (3.5) is a consequence of the following remark.

Remark. We observe that (1.14) implies $\operatorname{ord}_p \pi^{Mb(e-1)} < 1$.

Theorem 3.6. *Fix a positive integer e, let $m \in \{0, 1, \ldots, n\}$, and let b be a rational number satisfying* (1.14). *Let $\{\phi_i\}_{i \in I} \subseteq \Omega^m_{\mathbf{F}_q[x]/\mathbf{F}_q}$, I an index set, be a set of m-forms such that for each $r \ge 0$, the classes $\{[\phi_i]\}_{\deg \phi_i = r}$ form a basis for $E_e^{r,m-r}$ as an $\mathbf{F}_q$-vector space. Then their normalized Teichmüller liftings $\{\hat{\phi}_i\}_{i \in I} \subseteq \hat{F}^0\Omega^m_{C(b)}$ have the property that for each $r \in \mathbf{Z}$, the classes $\{[\pi^r \hat{\phi}_i]\}_{i \in I}$ form a basis for $\hat{E}^{r,m-r}_{Mb(e-1)+1}$ as an $\mathbf{F}_q$-vector space.*

Proof. The theorem asserts that if $\omega \in \hat{Z}^{r,m-r}_{Mb(e-1)+1}$, then there exist a finite subset $I_0 \subseteq I$, a collection $\{c_i\}_{i \in I_0} \subseteq \mathcal{O}_{\tilde{\Omega}_0}$ uniquely determined mod π, and

$$\xi \in \hat{Z}^{r-Mb(e-1),m-r-1+Mb(e-1)}_{Mb(e-1)}$$

such that

$$\omega \equiv \sum_{i \in I_0} c_i \pi^r \hat{\phi}_i + D(\xi) \pmod{\hat{F}^{r+1}\Omega^m_{C(b)}}.$$

Since $\pi^r \hat{F}^0 = \hat{F}^r$, we may reduce to the case $r = 0$ by multiplication by π^{-r}. So let $\omega \in \hat{F}^0\Omega^m_{C(b)}$ be such that

$$D(\omega) \in \hat{F}^{Mb(e-1)+1}\Omega^{m+1}_{C(b)}. \tag{3.7}$$

We must show that there exist $\{c_i\}_{i \in I_0} \subseteq \mathcal{O}_{\tilde{\Omega}_0}$ uniquely determined mod π and $\xi \in \hat{F}^{-Mb(e-1)}\Omega^{m-1}_{C(b)}$ such that

$$\omega \equiv \sum_{i \in I_0} c_i \hat{\phi}_i + D(\xi) \pmod{\hat{F}^1\Omega^m_{C(b)}}. \tag{3.8}$$

In view of (3.2), (3.7) is equivalent to the condition

$$\left(\sum_{j=0}^{e-1} \pi^{Mb(\delta-1)} d\hat{f}^{(\delta-j)} \right) \wedge \omega \in \hat{F}^{Mb(e-1)+1}\Omega^{m+1}_{C(b)} \tag{3.9}$$

and (3.8) is equivalent to the condition

$$\omega \equiv \sum_{i \in I_0} c_i \hat{\phi}_i + \left(\sum_{j=0}^{e-1} \pi^{Mb(\delta-1)} d\hat{f}^{(\delta-j)} \right) \wedge \xi \pmod{\hat{F}^1\Omega^m_{C(b)}}. \tag{3.10}$$

This reduces the proof of Theorem 3.6 to showing that, given $\omega \in \hat{F}^0\Omega^m_{C(b)}$ satisfying (3.9), there exist a finite set $\{c_i\}_{i\in I_0} \subseteq \mathcal{O}_{\tilde{\Omega}_0}$, uniquely determined mod π, and $\xi \in \hat{F}^{-Mb(e-1)}\Omega^{m-1}_{C(b)}$ satisfying (3.10).

We examine what (3.9) says about the coefficients of ω. Write

$$\omega = \sum_{k=0}^{\infty} \pi^{Mbk}\omega^{(k)}, \tag{3.11}$$

where $\omega^{(k)}$ is an m-form whose coefficients are homogeneous polynomials of degree k with coefficients in $\mathcal{O}_{\tilde{\Omega}_0}$. Since $\tilde{\Omega}_0 = K(\pi)$, where π is an $(M\delta)$-th root of p, we have the decomposition $\mathcal{O}_{\tilde{\Omega}_0} = \bigoplus_{l=0}^{M\delta-1} \pi^l \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers of K. This leads to a corresponding decomposition

$$\omega^{(k)} = \sum_{l=0}^{M\delta-1} \pi^l \omega^{(k)}_l, \tag{3.12}$$

where $\omega^{(k)}_l$ is an m-form whose coefficients are homogeneous polynomials of degree k with coefficients in $\mathcal{O}_K$. Substituting (3.12) into (3.11) and then substituting the result into (3.9) gives

$$\sum_{j=0}^{e-1}\sum_{l=0}^{M\delta-1}\sum_{k=0}^{\infty} \pi^{Mb(\delta-1+k)+l} d\hat{f}^{(\delta-j)} \wedge \omega^{(k)}_l \in \hat{F}^{Mb(e-1)+1}\Omega^{m+1}_{C(b)}. \tag{3.13}$$

We now fix k and consider the terms in this expression whose coefficients are homogeneous polynomials of degree k. We get

$$\sum_{j=0}^{e-1}\sum_{l=0}^{M\delta-1} \pi^{Mb(k+j)+l} d\hat{f}^{(\delta-j)} \wedge \omega^{(k-\delta+j+1)}_l \equiv 0 \pmod{\pi^{Mb(k+e-1)+1}}.$$

Cancel a factor of π^{Mbk} and group terms in the sum according to the power of π they contain:

$$\sum_{u=0}^{Mb(e-1)+M\delta-1} \pi^u \sum_{j,l\,:\,Mbj+l=u} d\hat{f}^{(\delta-j)} \wedge \omega^{(k-\delta+j+1)}_l \equiv 0 \pmod{\pi^{Mb(e-1)+1}}. \tag{3.14}$$

Lemma 3.15. *For $u = 0, 1, \ldots, Mb(e-1)$,*

$$\sum_{j,l\,:\,Mbj+l=u} d\hat{f}^{(\delta-j)} \wedge \omega^{(k-\delta+j+1)}_l \equiv 0 \pmod{p}.$$

Proof. Suppose that we have proved the congruence for all $u < u_0$ for some u_0, $0 \le u_0 \le Mb(e-1)$. We prove it for u_0. Since the left-hand side has coefficients in $\mathcal{O}_K$ and K is an unramified extension of $\mathbf{Q}_p$, it suffices to prove the congruence holds mod π. By the remark preceding Theorem 3.6, p is divisible by $\pi^{Mb(e-1)+1}$. So the congruence of the lemma holds mod $\pi^{Mb(e-1)+1}$ for all $u < u_0$. It then follows from (3.14) that

$$\sum_{u=u_0}^{Mb(e-1)+M\delta-1} \pi^u \sum_{j,l\,:\,Mbj+l=u} d\hat{f}^{(\delta-j)} \wedge \omega^{(k-\delta+j+1)}_l \equiv 0 \pmod{\pi^{Mb(e-1)+1}}.$$

Dividing by π^{u_0}, we conclude immediately that

$$\sum_{j,l\,:\,Mbj+l=u_0} d\hat{f}^{(\delta-j)} \wedge \omega_l^{(k-\delta+j+1)} \equiv 0 \pmod{\pi}. \qquad \square$$

We express Lemma 3.15 in a more convenient form. For $u = 0, 1, \ldots, Mb(e-1)$, write $u = MbJ + L$, with $0 \leq J \leq e-1$ and $0 \leq L < Mb$. Rewriting the sum in Lemma 3.15 in terms of J and L and replacing k by $k + \delta - J - 1$, we get the following.

Corollary 3.16. *For $0 \leq J < e-1$ and $0 \leq L < Mb$ or $J = e-1$ and $L = 0$,*

$$\sum_{j=0}^{J} d\hat{f}^{(\delta-j)} \wedge \omega_{L+Mb(J-j)}^{(k+j-J)} \equiv 0 \pmod{p}.$$

Put

$$\omega_k = \omega_0^{(k)} + \omega_{Mb}^{(k-1)} + \cdots + \omega_{Mb(e-1)}^{(k-e+1)} \in \Omega_{C(b)}^m$$

and let $\bar{\omega}_k \in \Omega^m_{\mathbf{F}_q[x]/\mathbf{F}_q}$ be its reduction mod π. Taking $L = 0$ and $J = 0, 1, \ldots, e-1$ in Corollary 3.16 shows that $\bar{\omega}_k \in Z_e^{r,m-r}$ with $r = k + (n-m)(\delta - 1)$. Put

$$I^{(k)} = \{i \in I \mid \deg_{\text{coeff}} \phi_i = k\}.$$

The definition of the ϕ_i implies that there exist $\{\bar{c}_i\}_{i\in I^{(k)}} \subseteq \mathbf{F}_q$ and $\{\bar{\xi}_j^{(k-\delta+j)}\}_{j=1}^{e} \subseteq \Omega^{m-1}_{\mathbf{F}_q[x]/\mathbf{F}_q}$, with the coefficients of $\bar{\xi}_j^{(k-\delta+j)}$ being homogeneous polynomials of degree $k - \delta + j$, such that

$$\bar{\omega}_k = \sum_{i\in I^{(k)}} \bar{c}_i\phi_i + \sum_{j=0}^{e-1} df^{(\delta-j)} \wedge \bar{\xi}_{j+1}^{(k-\delta+j+1)} + (\text{terms of degree} < k) \tag{3.17}$$

and such that

$$\sum_{j=0}^{i} df^{(\delta-j)} \wedge \bar{\xi}_{j+e-i}^{(k-\delta+j+e-i)} = 0 \tag{3.18}$$

for $i = 0, 1, \ldots, e-2$.

Let $c_i \in \mathcal{O}_K$ be the Teichmüller lifting of $\bar{c}_i \in \mathbf{F}_q$ and let $\xi_j^{(k+j-\delta)} \in \Omega_{C(b)}^{m-1}$ be the nonnormalized Teichmüller lifting of $\bar{\xi}_j^{(k+j-\delta)}$. Thus $\xi_j^{(k+j-\delta)}$ is an $(m-1)$-form whose coefficients are homogeneous polynomials of degree $k+j-\delta$ with coefficients that are units in $\mathcal{O}_K$. Then (3.17) implies

$$\pi^{Mbk}\omega_k \equiv \sum_{i\in I^{(k)}} c_i\hat{\phi}_i + \sum_{j=0}^{e-1} \pi^{Mbk} d\hat{f}^{(\delta-j)} \wedge \xi_{j+1}^{(k-\delta+j+1)} \pmod{\hat{F}^1\Omega_{C(b)}^m} \tag{3.19}$$

and (3.18) implies

$$\sum_{j=0}^{i} d\hat{f}^{(\delta-j)} \wedge \xi_{j+e-i}^{(k-\delta+j+e-i)} \equiv 0 \pmod{p} \tag{3.20}$$

for $i = 0, 1, \ldots, e-2$.

By the definition of $C(b)$, there exists a positive integer k_ω such that $\pi^{Mbk}\omega^{(k)} \equiv 0 \pmod{\hat{F}^1\Omega^m_{C(b)}}$ for all $k > k_\omega$. Let $I_0 = \bigcup_{k\le k_\omega} I^{(k)}$, a finite subset of I. Put

$$\xi_i = \sum_{k=0}^{\infty} \pi^{Mb(k-\delta+i)}\xi_i^{(k-\delta+i)} \in \hat{F}^0\Omega^{m-1}_{C(b)}$$

for $i = 1, 2, \ldots, e$. Then (3.11), (3.19), and the observation that $\pi^{Mbk}\omega^{(k)} \equiv \pi^{Mbk}\omega_k \pmod{\hat{F}^1\Omega^m_{C(b)}}$ imply

$$\omega \equiv \sum_{i\in I_0} c_i\hat{\phi}_i + \sum_{j=0}^{e-1} \pi^{Mb(\delta-j-1)} d\hat{f}^{(\delta-j)} \wedge \xi_{j+1} \pmod{\hat{F}^1\Omega^m_{C(b)}}. \tag{3.21}$$

Also, since p is divisible by $\pi^{Mb(e-1)+1}$, (3.20) implies

$$\sum_{j=0}^{i} \pi^{Mb(\delta-j-1)} d\hat{f}^{(\delta-j)} \wedge \xi_{j+e-i} \equiv 0 \pmod{\hat{F}^{Mb(e-1)+1}\Omega^m_{C(b)}} \tag{3.22}$$

for $i = 0, 1, \ldots, e-2$.

Now put

$$\xi = \sum_{j=0}^{e-1} \pi^{-Mbj}\xi_{j+1} \in \hat{F}^{-Mb(e-1)}\Omega^{m-1}_{C(b)}. \tag{3.23}$$

With this choice of ξ, the right-hand side of (3.10) becomes

$$\sum_{i\in I_0} c_i\hat{\phi}_i + \left(\sum_{i=0}^{e-1} \pi^{Mbi}\big(\pi^{Mb(\delta-i-1)} d\hat{f}^{(\delta-i)}\big)\right) \wedge \left(\sum_{j=0}^{e-1} \pi^{-Mbj}\xi_{j+1}\right). \tag{3.24}$$

When (3.24) is expanded, the wedge product of a pair of terms with $i > j$ lies in $\hat{F}^{Mb}\Omega^m_{C(b)}$. Putting $k = j - i$ when $j \ge i$, we see that (3.24) is congruent mod $\hat{F}^{Mb}\Omega^m_{C(b)}$ to

$$\sum_{i\in I_0} c_i\hat{\phi}_i + \sum_{k=0}^{e-1} \pi^{-Mbk} \sum_{l=0}^{e-1-k} \pi^{Mb(\delta-l-1)} d\hat{f}^{(\delta-l)} \wedge \xi_{l+k+1}. \tag{3.25}$$

For $k = 1, \ldots, e-1$, the corresponding summand of (3.25) lies in $\hat{F}^1\Omega^m_{C(b)}$ by (3.22). Hence (3.25) is congruent mod $\hat{F}^1\Omega^m_{C(b)}$ to

$$\sum_{i\in I_0} c_i\hat{\phi}_i + \sum_{l=0}^{e-1} \pi^{Mb(\delta-l-1)} d\hat{f}^{(\delta-l)} \wedge \xi_{l+1}.$$

But this is $\equiv \omega \pmod{\hat{F}^1\Omega^m_{C(b)}}$ by (3.21). Thus congruence (3.10) holds.

To complete the proof of Theorem 3.6, it remains to show that the c_i are uniquely determined mod π. Suppose there exist a finite subset $I_0 \subseteq I$, $\{c_i\}_{i\in I_0} \subseteq \mathcal{O}_{\tilde{\Omega}_0}$ and $\xi \in \hat{F}^{-Mb(e-1)}\Omega^{m-1}_{C(b)}$ such that

$$\sum_{i\in I_0} c_i\hat{\phi}_i \equiv \sum_{j=0}^{e-1} \pi^{Mb(\delta-1)} d\hat{f}^{(\delta-j)} \wedge \xi \pmod{\hat{F}^1\Omega^m_{C(b)}}. \tag{3.26}$$

We must show that $c_i \equiv 0 \pmod{\pi}$ for all $i \in I_0$. Write

$$\xi = \pi^{-Mb(e-1)} \sum_{k=0}^{\infty} \pi^{Mbk} \xi^{(k)}, \tag{3.27}$$

where the coefficients of $\xi^{(k)} \in \Omega^{m-1}_{C(b)}$ are homogeneous polynomials of degree k with coefficients in $\mathcal{O}_{\tilde{\Omega}_0}$. As in (3.12), we write

$$\xi^{(k)} = \sum_{l=0}^{M\delta-1} \pi^l \xi_l^{(k)}, \tag{3.28}$$

where $\xi_l^{(k)}$ is an $(m-1)$-form whose coefficients are homogeneous polynomials of degree k with coefficients in $\mathcal{O}_K$. Substituting (3.28) into (3.27) and substituting the result into (3.26) gives (after multiplication by $\pi^{Mb(e-1)}$)

$$\pi^{Mb(e-1)} \sum_{i\in I_0} c_i \hat{\phi}_i \equiv \sum_{j=0}^{e-1} \sum_{l=0}^{M\delta-1} \sum_{k=0}^{\infty} \pi^{Mb(\delta-1+k)+l} d\hat{f}^{(\delta-j)} \wedge \xi_l^{(k)} \pmod{\hat{F}^{Mb(e-1)+1}\Omega^m_{C(b)}}. \tag{3.29}$$

We now fix k and consider the terms with coefficients of degree k in this equation. If $\deg_{\text{coeff}} \phi_i < k$, then $\hat{\phi}_i$ contains no terms with coefficients of degree k. If $\deg_{\text{coeff}} \phi_i > k$, then the terms in $\hat{\phi}_i$ with coefficients of degree k lie in $\hat{F}^1\Omega^m_{C(b)}$. We thus obtain

$$\pi^{Mb(e-1)} \sum_{i\in I^{(k)}} c_i \hat{\phi}_i \equiv \sum_{j=0}^{e-1} \sum_{l=0}^{M\delta-1} \pi^{Mb(k+j)+l} d\hat{f}^{(\delta-j)} \wedge \xi_l^{(k-\delta+j+1)} \pmod{\hat{F}^{Mb(e-1)+1}\Omega^m_{C(b)}}. \tag{3.30}$$

Let $\tilde{\phi}_i$ be the nonnormalized Teichmüller lifting of ϕ_i. Thus $\hat{\phi}_i = \pi^{Mbk}\tilde{\phi}_i$ and the reduction of $\tilde{\phi}_i$ mod π equals ϕ_i. Cancelling a factor of π^{Mbk} in (3.30) and grouping the terms on the right-hand side according to the power of π that they contain gives

$$\pi^{Mb(e-1)} \sum_{i\in I^{(k)}} c_i \tilde{\phi}_i \equiv \sum_{u=0}^{Mb(e-1)+M\delta-1} \pi^u \sum_{j,l\,:\,Mbj+l=u} d\hat{f}^{(\delta-j)} \wedge \xi_l^{(k-\delta+j+1)} \pmod{\pi^{Mb(e-1)+1}}. \tag{3.31}$$

Lemma 3.32. *For $u = 0, 1, \ldots, Mb(e-1)-1$,*

$$\sum_{j,l\,:\,Mbj+l=u} d\hat{f}^{(\delta-j)} \wedge \xi_l^{(k-\delta+j+1)} \equiv 0 \pmod{p}$$

and

$$\sum_{j,l\,:\,Mbj+l=Mb(e-1)} d\hat{f}^{(\delta-j)} \wedge \xi_l^{(k-\delta+j+1)} \equiv \sum_{i\in I^{(k)}} c_i\tilde{\phi}_i \pmod{\pi}.$$

Proof. The first congruence is proved by induction on u exactly as in Lemma 3.15, using the fact that the left-hand side of (3.31) is $\equiv 0 \bmod \pi^{Mb(e-1)}$. Using the result of the first congruence in (3.31) then gives

$$\pi^{Mb(e-1)} \sum_{i\in I^{(k)}} c_i\tilde{\phi}_i \equiv \sum_{u=Mb(e-1)}^{Mb(e-1)+M\delta-1} \pi^u \sum_{j,l\,:\,Mbj+l=u} d\hat{f}^{(\delta-j)} \wedge \xi_l^{(k-\delta+j+1)} \pmod{\pi^{Mb(e-1)+1}},$$

from which the second congruence follows after cancelling $\pi^{Mb(e-1)}$. □

Writing $u = MbJ + L$ with $0 \le J < e-1$ and $0 \le L < Mb$ and replacing k by $k+e-J-1$, we can reformulate Lemma 3.32 as the following.

Corollary 3.33. *For $0 \le J < e-1$ and $0 \le L < Mb$,*

$$\sum_{j=0}^{J} d\hat{f}^{(\delta-j)} \wedge \xi_{L+Mb(J-j)}^{(k-\delta+e+j-J)} \equiv 0 \pmod{p}$$

and

$$\sum_{j=0}^{e-1} d\hat{f}^{(\delta-j)} \wedge \xi_{Mb(e-1-j)}^{(k-\delta+j+1)} \equiv \sum_{i\in I^{(k)}} c_i\tilde{\phi}_i \pmod{\pi}.$$

Let $\bar{c}_i \in \mathbf{F}_q$ be the reduction mod π of c_i and let $\bar{\xi}_{L+Mb(J-j)}^{(k-\delta+e+j-J)}$ be the reduction mod π of $\xi_{L+Mb(J-j)}^{(k-\delta+e+j-J)}$. Taking $L=0$ in Corollary 3.33 shows that

$$\sum_{i\in I^{(k)}} \bar{c}_i\phi_i = \sum_{j=0}^{e-1} df^{(\delta-j)} \wedge \bar{\xi}_{Mb(e-1-j)}^{(k-\delta+j+1)} + (\text{terms of degree} < k)$$

and that for $J = 0, 1, \dots, e-2$,

$$\sum_{j=0}^{J} df^{(\delta-j)} \wedge \bar{\xi}_{Mb(J-j)}^{(k-\delta+e+j-J)} = 0,$$

i.e., $[\sum_{i\in I^{(k)}} \bar{c}_i\phi_i] = 0$ in $E_e^{r,m-r}$, where $r = k+(n-m)(\delta-1)$. Since $\{\phi_i\}_{i\in I^{(k)}}$ is a basis for $E_e^{r,m-r}$, we conclude that $\bar{c}_i = 0$ for all $i \in I^{(k)}$. Since k was arbitrary, this shows that $c_i \equiv 0 \pmod{\pi}$ for all i. This completes the proof of Theorem 3.6.

4. Computation of p-adic cohomology

We derive some corollaries to Theorem 3.6.

Corollary 4.1. *Suppose there exist e, m such that $E_e^{r,m-r} = 0$ for all $r \ge 0$. Then for every rational number b satisfying (1.14), we have $H^m(\Omega^{\cdot}_{C(b)}, D) = 0$.*

Proof. Let $\omega \in \Omega^m_{C(b)}$ satisfy $D(\omega) = 0$. Without loss of generality, we may assume $\omega \in \hat{F}^0\Omega^m_{C(b)}$. By Theorem 3.6, $\hat{E}^{r,m-r}_{Mb(e-1)+1} = 0$ for all $r \in \mathbf{Z}$. So there exist

$$\omega_0 \in \hat{F}^0\Omega^m_{C(b)}, \quad \xi_0 \in \hat{F}^{-Mb(e-1)}\Omega^{m-1}_{C(b)}$$

such that

$$\omega = \pi\omega_0 + D(\xi_0). \tag{4.2}$$

Suppose that for some $t \ge 0$ we have found

$$\omega_t \in \hat{F}^0\Omega^m_{C(b)}, \quad \xi_t \in \hat{F}^{-Mb(e-1)}\Omega^{m-1}_{C(b)}$$

such that

$$\omega = \pi^{t+1}\omega_t + D(\xi_t) \tag{4.3}$$

and such that

$$\xi_t - \xi_{t-1} \in \hat{F}^{-Mb(e-1)+t}\Omega^{m-1}_{C(b)}.$$

Equation (4.3) implies $D(\omega_t) = 0$. So we may apply Theorem 3.6 to ω_t to conclude that there exist

$$\omega_{t+1} \in \hat{F}^0\Omega^m_{C(b)}, \quad \xi'_t \in \hat{F}^{-Mb(e-1)}\Omega^{m-1}_{C(b)}$$

such that

$$\omega_t = \pi\omega_{t+1} + D(\xi'_t). \tag{4.4}$$

Put $\xi_{t+1} = \xi_t + \pi^{t+1}\xi'_t$. Substituting (4.4) into (4.3) gives

$$\omega = \pi^{t+2}\omega_{t+1} + D(\xi_{t+1}) \tag{4.5}$$

with

$$\xi_{t+1} - \xi_t \in \hat{F}^{-Mb(e-1)+t+1}\Omega^{m-1}_{C(b)}.$$

It follows that $\{\xi_t\}_{t=0}^{\infty}$ converges to an element $\xi \in \hat{F}^{-Mb(e-1)}\Omega^{m-1}_{C(b)}$ satisfying

$$\omega = D(\xi).$$

This establishes the corollary. □

Corollary 4.6. *Let e be a positive integer and b a rational number satisfying* (1.14). *Suppose there is a finite index set I and n-forms $\{\phi_i\}_{i\in I} \subseteq \Omega^n_{\mathbf{F}_q[x]/\mathbf{F}_q}$ such that for each $r \geq 0$ the classes $\{\phi_i\}_{\deg\phi_i = r}$ form a basis for $E_e^{r,n-r}$ as an $\mathbf{F}_q$-vector space. Suppose also that for all $r \in \mathbf{Z}$,*

$$\hat{E}^{r,n-r}_{Mb(e-1)+1} = \hat{E}^{r,n-r}_{Mb(e-1)+2} = \cdots. \tag{4.7}$$

Then $\{[\hat{\phi}_i]\}_{i\in I}$ is a basis for $H^n(\Omega^{\cdot}_{C(b)}, D)$ (as an $\tilde{\Omega}_0$-vector space).

Proof. Let $\omega \in \Omega^n_{C(b)}$. Without loss of generality, we may assume $\omega \in \hat{F}^0\Omega^n_{C(b)}$. By Theorem 3.6, there exist

$$\{c_i^{(0)}\}_{i\in I} \subseteq \mathcal{O}_{\tilde{\Omega}_0}, \quad \omega_0 \in \hat{F}^0\Omega^n_{C(b)}, \quad \xi_0 \in \hat{F}^{-Mb(e-1)}\Omega^{n-1}_{C(b)}$$

such that

$$\omega = \pi\omega_0 + \sum_{i\in I} c_i^{(0)}\hat{\phi}_i + D(\xi_0). \tag{4.8}$$

Suppose that for some $t \geq 0$ we have found

$$\{c_i^{(t)}\}_{i\in I} \subseteq \mathcal{O}_{\tilde{\Omega}_0}, \quad \omega_t \in \hat{F}^0\Omega^n_{C(b)}, \quad \xi_t \in \hat{F}^{-Mb(e-1)}\Omega^{n-1}_{C(b)}$$

such that

$$\omega = \pi^{t+1}\omega_t + \sum_{i\in I} c_i^{(t)}\hat{\phi}_i + D(\xi_t) \tag{4.9}$$

and such that

$$c_i^{(t)} - c_i^{(t-1)} \in \pi^t\mathcal{O}_{\tilde{\Omega}_0}, \quad \xi_t - \xi_{t-1} \in \hat{F}^{-Mb(e-1)+t}\Omega^{n-1}_{C(b)}.$$

Applying Theorem 3.6 to ω_t, we see that there exist

$$\{\tilde{c}_i^{(t)}\}_{i\in I} \subseteq \mathcal{O}_{\tilde{\Omega}_0}, \quad \omega_{t+1} \in \hat{F}^0\Omega^n_{C(b)}, \quad \xi'_t \in \hat{F}^{-Mb(e-1)}\Omega^{n-1}_{C(b)}$$

such that

$$\omega_t = \pi\omega_{t+1} + \sum_{i\in I} \tilde{c}_i^{(t)}\hat{\phi}_i + D(\xi'_t). \tag{4.10}$$

Put $\xi_{t+1} = \xi_t + \pi^{t+1}\xi'_t$ and $c_i^{(t+1)} = c_i^{(t)} + \pi^{t+1}\tilde{c}_i^{(t)}$. Substituting (4.10) into (4.9) gives

$$\omega = \pi^{t+2}\omega_{t+1} + \sum_{i\in I} c_i^{(t+1)}\hat{\phi}_i + D(\xi_{t+1}) \tag{4.11}$$

with

$$c_i^{(t+1)} - c_i^{(t)} \in \pi^{t+1}\mathcal{O}_{\tilde{\Omega}_0}, \quad \xi_{t+1} - \xi_t \in \hat{F}^{-Mb(e-1)+t+1}\Omega^{n-1}_{C(b)}.$$

It follows that $\{\xi_t\}_{t=0}^\infty$ converges to an element $\xi \in \hat{F}^{-Mb(e-1)}\Omega^{n-1}_{C(b)}$ and $\{c_i^{(t)}\}_{t=0}^\infty$ converges to an element $c_i \in \mathcal{O}_{\tilde{\Omega}_0}$ for $i \in I$ satisfying

$$\omega = \sum_{i\in I} c_i\hat{\phi}_i + D(\xi).$$

This shows that $\{[\hat{\phi}_i]\}_{i\in I}$ spans $H^n(\Omega^{\boldsymbol{\cdot}}_{C(b)}, D)$.

Suppose we had a relation

$$\sum_{i\in I} c_i\hat{\phi}_i = D(\xi), \tag{4.12}$$

where $\{c_i\}_{i\in I} \subseteq \tilde{\Omega}_0$ and $\xi \in \Omega^{n-1}_{C(b)}$. If the c_i were not all zero, then after multiplication by a suitable power of π we may assume that $c_i \in \mathcal{O}_{\tilde{\Omega}_0}$ for all i and $c_i \notin \pi\mathcal{O}_{\tilde{\Omega}_0}$ for some i. Thus the left-hand side of (4.12) lies in $\hat{F}^0\Omega^n_{C(b)}$ but not in $\hat{F}^1\Omega^n_{C(b)}$. Since the filtration $\hat{F}^{\boldsymbol{\cdot}}$ is exhaustive, there exists $r \geq 0$ such that

$$\xi \in \hat{F}^{-Mb(e-1)-r}\Omega^{m-1}_{C(b)}.$$

Equation (4.12) then says that $[\sum_{i\in I} c_i\hat{\phi}_i] = 0$ in $\hat{E}^{0,n}_{Mb(e-1)+r+1}$. By (4.7), we have $[\sum_{i\in I} c_i\hat{\phi}_i] = 0$ in $\hat{E}^{0,n}_{Mb(e-1)+1}$. Theorem 3.6 now implies that $c_i \equiv 0 \pmod{\pi}$ for all i, a contradiction. Thus $c_i = 0$ for all i. □

Proposition 4.13. *Suppose there exist e, m such that $E_e^{r,m-1-r} = E_e^{r,m+1-r} = 0$ for all $r \geq 0$. Then for every rational number b satisfying* (1.14)*,*

$$\hat{E}^{r,m-r}_{Mb(e-1)+1} = \hat{E}^{r,m-r}_{Mb(e-1)+2} = \cdots$$

for all $r \in \mathbf{Z}$.

Proof. Theorem 3.6 and the hypothesis of the proposition imply that

$$\hat{E}^{r,m-1-r}_{Mb(e-1)+1} = \hat{E}^{r,m+1-r}_{Mb(e-1)+1} = 0$$

for all $r \in \mathbf{Z}$. The conclusion then follows from general properties of spectral sequences. □

Proof of Theorem 1.13. The first assertion of Theorem 1.13 follows from Corollary 4.1. Suppose that (1.11) holds. The discussion in the introduction shows that (1.11) implies

$$\dim_{\mathbf{F}_q}\left(\bigoplus_{r=0}^{\infty} E_e^{r,n-r}\right) = M_f.$$

By Proposition 4.13 with $m = n$, we can apply Corollary 4.6 to conclude that

$$\dim_{\tilde{\Omega}_0} H^n(\Omega^{\cdot}_{C(b)}, D) = M_f. \qquad \square$$

5. Proof of Theorem 1.18

In this section we will always be working in the case of characteristic p. In particular, the notation "$\hat{f}_i$" means not the Teichmüller lifting of f_i but rather that the factor f_i is to be omitted from a product of similar factors. Throughout this section, we assume the hypothesis of Theorem 1.18. Put

$$Z^k = \{\omega \in \Omega^k_{\mathbf{F}_q[x]/\mathbf{F}_q} \mid df^{(\delta)} \wedge \omega = 0\}.$$

We leave it to the reader to check from the definition of the spectral sequence (1.6) that the conclusion of Theorem 1.18 is equivalent to the following assertion.

Proposition 5.1. *Let $k < n$ and let $\omega \in Z^k$ be a homogeneous form such that*

$$df^{(\delta')} \wedge \omega = df^{(\delta)} \wedge \xi \tag{5.2}$$

for some homogeneous form $\xi \in \Omega^k_{\mathbf{F}_q[x]/\mathbf{F}_q}$. Then there exist homogeneous forms $\eta_1 \in Z^{k-1}$, $\eta_2 \in \Omega^{k-1}_{\mathbf{F}_q[x]/\mathbf{F}_q}$, such that

$$\omega = df^{(\delta')} \wedge \eta_1 + df^{(\delta)} \wedge \eta_2. \tag{5.3}$$

Before starting the proof, we observe that Kita [9] has, in effect, characterized the elements of Z^k. For notational convenience, we define a 1-form $\Theta \in \Omega^1_{\mathbf{F}_q[x]/\mathbf{F}_q}$ by

$$\Theta = f_1 \cdots f_r \sum_{i=1}^{r} a_i \frac{df_i}{f_i} \quad \left(= f_1 \cdots f_r \frac{df^{(\delta)}}{f^{(\delta)}} \right),$$

and for any l-tuple $1 \le i_1 < \cdots < i_l \le r$ we define

$$\Omega_{i_1 \cdots i_l} = \frac{df_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{df_{i_l}}{f_{i_l}},$$

a rational l-form with logarithmic poles along the divisor $f_1 \cdots f_r = 0$ in $\mathbf{A}^n$. Note that $\Theta \wedge \Omega_{i_1 \cdots i_l}$ has polynomial coefficients, hence lies in $\Omega^{l+1}_{\mathbf{F}_q[x]/\mathbf{F}_q}$.

Proposition 5.4. *Let $k < n$ and let $\omega \in Z^k$ be homogeneous. Then*

$$\omega = \Theta \wedge \left(\sum_{l=0}^{k-1} \sum_{1 \le i_1 < \cdots < i_l \le r} \Omega_{i_1 \cdots i_l} \wedge \alpha_{i_1 \cdots i_l} \right) \tag{5.5}$$

for some homogeneous forms $\alpha_{i_1 \cdots i_l} \in \Omega^{k-1-l}_{\mathbf{F}_q[x]/\mathbf{F}_q}$.

Proof. The Euler relation and the hypothesis $(p, \delta) = 1$ imply that $f^{(\delta)}$ lies in the ideal generated by the $\partial f^{(\delta)}/\partial x_i$. The equation $df^{(\delta)} \wedge \omega = 0$ then implies, by a standard result ([10, Theorem 16.4]), that there exists $\alpha \in \Omega^{k-1}_{\mathbf{F}_q[x]/\mathbf{F}_q}$ such that

$$f^{(\delta)} \omega = df^{(\delta)} \wedge \alpha, \tag{5.6}$$

or, equivalently,

$$f_1 \cdots f_r \omega = \Theta \wedge \alpha. \tag{5.7}$$

This implies that $a_i f_1 \cdots \hat{f}_i \cdots f_r \, df_i \wedge \alpha \in (f_i)$ for $i = 1, \dots, r$. But $(p, a_i) = 1$, and our hypothesis implies that f_j, f_i form a regular sequence for $j \neq i$; hence $df_i \wedge \alpha \in (f_i)$ for $i = 1, \dots, r$. It then follows from [9, Proposition 2.2.3] that

$$\alpha = f_1 \cdots f_r \sum_{l=0}^{k-1} \sum_{1 \leq i_1 < \cdots < i_l \leq r} \Omega_{i_1 \cdots i_l} \wedge \alpha_{i_1 \cdots i_l} \tag{5.8}$$

for some homogeneous forms $\alpha_{i_1 \cdots i_l} \in \Omega^{k-1-l}_{\mathbf{F}_q[x]/\mathbf{F}_q}$. (Although Kita works over $\mathbf{C}$, his proof is valid for any field.) Substituting (5.8) into (5.7) gives the proposition. □

The following lemma is the main technical tool for the proof of Proposition 5.1.

Lemma 5.9. *Suppose that* $\{i_1, \dots, i_l\}$ *is a nonempty subset of* $\{1, \dots, r\}$ *with either* $l \geq 2$ *or* $l = 1$ *and* $a_{i_1} > 1$ *and that* $\beta \in \Omega^m_{\mathbf{F}_q[x]/\mathbf{F}_q}$ *is a homogeneous* m*-form with* $m \leq n - l - 1$ *such that*

$$df^{(\delta')} \wedge df_{i_1} \wedge \cdots \wedge df_{i_l} \wedge \beta \equiv 0 \pmod{(f_{i_1}, \dots, f_{i_l})}.$$

Then there exist homogeneous $(m-1)$*-forms* β_j *for* $j = 0, 1, \dots, l$ *and* β'_j *for* $j = 1, \dots, l$ *such that*

$$\beta = df^{(\delta')} \wedge \beta_0 + \sum_{j=1}^{l} df_{i_j} \wedge \beta_j + \sum_{j=1}^{l} f_{i_j} \beta'_j.$$

Proof. Consider the expansion of $df^{(\delta')} \wedge df_{i_1} \wedge \cdots \wedge df_{i_l}$ relative to the basis for $\Omega^{l+1}_{\mathbf{F}_q[x]/\mathbf{F}_q}$ consisting of the $(l+1)$-fold exterior products of the dx_i, $i = 1, \dots, n$. By the theorem of [12], it suffices to show that the coefficients in this expansion generate an ideal I of depth $n - l$ in the quotient ring $\mathbf{F}_q[x]/(f_{i_1}, \dots, f_{i_l})$, i.e., the only maximal ideal containing I is the one generated by $x_1, \dots, x_n$. Suppose there were some other maximal ideal $\mathbf{m}$ containing I. Then $\mathbf{m}$ would correspond to a point in $\mathbf{A}^n$, other than the origin, which is a common zero of $f_{i_1}, \dots, f_{i_l}$ and at which there is a linear relation between the differentials $df^{(\delta')}, df_{i_1}, \dots, df_{i_l}$. Since $f_{i_1} = \cdots = f_{i_l} = 0$ is a smooth complete intersection in $\mathbf{A}^n$ except at the origin, the differentials $df_{i_1}, \dots, df_{i_l}$ are independent. So $df^{(\delta')}$ must be a linear combination of $df_{i_1}, \dots, df_{i_l}$ at this point. But then the Euler relations for the polynomials $f^{(\delta')}, f_{i_1}, \dots, f_{i_l}$, together with the hypothesis that $(p, \delta') = 1$, imply that $f^{(\delta')}$ also vanishes at this point, contradicting the hypothesis that $f^{(\delta')} = f_{i_1} = \cdots = f_{i_l} = 0$ defines a smooth complete intersection in $\mathbf{A}^n$ except at the origin. □

Let $\omega \in Z^k$ and suppose that for some s, $1 \leq s \leq r$, and $b_s, \dots, b_r$, $1 \leq b_i \leq a_i$ for $i = s, \dots, r$, we have

$$\omega = f_1^{a_1} \cdots f_{s-1}^{a_{s-1}} f_s^{b_s} \cdots f_r^{b_r} \sum_{i=1}^{r} a_i \frac{df_i}{f_i} \wedge \left(\sum_{l=0}^{k-1} \sum_{1 \leq i_1 < \cdots < i_l \leq r} \Omega_{i_1 \cdots i_l} \wedge \alpha_{i_1 \cdots i_l} \right) \tag{5.10}$$

for some homogeneous forms $\alpha_{i_1 \cdots i_l} \in \Omega^{k-1-l}_{\mathbf{F}_q[x]/\mathbf{F}_q}$.

Lemma 5.11. *If $b_s < a_s$, then there exists $\eta \in Z^{k-1}$ such that*

$$\begin{aligned}(5.12)\quad & \omega - df^{(\delta')} \wedge \eta \\ & = f_1^{a_1} \cdots f_{s-1}^{a_{s-1}} f_s^{b_s} \cdots f_r^{b_r} \sum_{i=1}^{r} a_i \frac{df_i}{f_i} \wedge \left(\sum_{l=0}^{k-1} \sum_{1 \le i_1 < \cdots < i_l \le r} \Omega_{i_1 \cdots i_l} \wedge \alpha'_{i_1 \cdots i_l} \right),\end{aligned}$$

where all $\alpha'_{i_1 \cdots i_l}$ are divisible by f_s.

Observe that Proposition 5.4 implies that every $\omega \in Z^k$ can be written in the form (5.10) with $s = 1$ and $b_1 = \cdots = b_r = 1$. When the conclusion of Lemma 5.11 holds, we can factor out f_s from each $\alpha'_{i_1 \cdots i_l}$ and replace b_s by $b_s + 1$ in (5.12). Induction on b_s and s then allows us to replace $b_s, \ldots, b_r$ by $a_s, \ldots, a_r$, respectively, in (5.12), giving the following.

Corollary 5.13. *If $\omega \in Z^k$ with $k < n$, then there exists $\eta \in Z^{k-1}$ such that*

$$\omega - df^{(\delta')} \wedge \eta = df^{(\delta)} \wedge \left(\sum_{l=0}^{k-1} \sum_{1 \le i_1 < \cdots < i_l \le r} \Omega_{i_1 \cdots i_l} \wedge \alpha_{i_1 \cdots i_l} \right)$$

for some homogeneous forms $\alpha_{i_1 \cdots i_l} \in \Omega^{k-1-l}_{\mathbf{F}_q[x]/\mathbf{F}_q}$.

Proof of Lemma 5.11. Put

$$\begin{aligned}(5.14)\quad \tilde{\omega} &= \omega / (f_1^{a_1 - 1} \cdots f_{s-1}^{a_{s-1}-1} f_s^{b_s - 1} \cdots f_r^{b_r - 1}) \\ &= \Theta \wedge \left(\sum_{l=0}^{k-1} \sum_{1 \le i_1 < \cdots < i_l \le r} \Omega_{i_1 \cdots i_l} \wedge \alpha_{i_1 \cdots i_l} \right).\end{aligned}$$

We show there exists $\tilde{\eta} \in Z^{k-1}$ such that

$$(5.15)\qquad \tilde{\omega} - df^{(\delta')} \wedge \tilde{\eta} = \Theta \wedge \left(\sum_{l=0}^{k-1} \sum_{1 \le i_1 < \cdots < i_l \le r} \Omega_{i_1 \cdots i_l} \wedge \alpha'_{i_1 \cdots i_l} \right),$$

where all $\alpha'_{i_1 \cdots i_l}$ are divisible by f_s. Lemma 5.11 follows from (5.15) by multiplication by $f_1^{a_1 - 1} \cdots f_{s-1}^{a_{s-1}-1} f_s^{b_s - 1} \cdots f_r^{b_r - 1}$.

Equation (5.2) implies

$$df^{(\delta')} \wedge \tilde{\omega} = f_s^{a_s - b_s} \cdots f_r^{a_r - b_r} \Theta \wedge \xi.$$

So if $b_s < a_s$ we have

$$(5.16)\qquad df^{(\delta')} \wedge \tilde{\omega} \equiv 0 \pmod{(f_s)}.$$

We prove the existence of $\tilde{\eta}$ by descending induction on l. Since $\alpha_{i_1 \cdots i_l} = 0$ for $l > k - 1$, we may assume that for some m, $0 \le m \le k - 1$, we have that $\alpha_{i_1 \cdots i_l}$ is divisible by f_s for all $l \ge m + 1$. We show that we can choose $\tilde{\eta}$ so that the $\alpha'_{i_1 \cdots i_l}$ are divisible by f_s for all $l \ge m$.

Fix an m-tuple $1 \le i_1 < \cdots < i_m \le r$ with $s \notin \{i_1, \ldots, i_m\}$, say, $i_t < s < i_{t+1}$. Expand the right-hand side of (5.14) using the definitions of Θ and $\Omega_{i_1 \cdots i_l}$. The term containing $df_{i_1} \wedge \cdots \wedge df_s \wedge \cdots \wedge df_{i_m}$ is

$$(5.17)\quad df_{i_1} \wedge \cdots \wedge df_s \wedge \cdots \wedge df_{i_m} \wedge f_1 \cdots \hat{f}_{i_1} \cdots \hat{f}_s \cdots \hat{f}_{i_m} \cdots f_r \Big((-1)^t a_s \alpha_{i_1 \cdots i_m} + \sum_{j=1}^{t} (-1)^{j-1} a_{i_j} \alpha_{i_1 \cdots \hat{\imath}_j \cdots s \cdots i_m} + \sum_{j=t+1}^{m} (-1)^j a_{i_j} \alpha_{i_1 \cdots s \cdots \hat{\imath}_j \cdots i_m} \Big).$$

Using the induction hypothesis that $\alpha_{i_1 \cdots i_l}$ is divisible by f_s for all $l \geq m+1$, one sees that the term containing any other product $df_{i_1} \wedge \cdots \wedge df_{i_l}$ (for $l \geq 0$) on the right-hand side of (5.14) lies in the ideal $(f_s, f_{i_1}, \ldots, f_{i_m})$. Equation (5.16) then implies

$$(5.18)\quad df^{(\delta')} \wedge df_{i_1} \wedge \cdots \wedge df_s \wedge \cdots \wedge df_{i_m} \wedge f_1 \cdots \hat{f}_{i_1} \cdots \hat{f}_s \cdots \hat{f}_{i_m} \cdots f_r \Big((-1)^t a_s \alpha_{i_1 \cdots i_m} + \sum_{j=1}^{t} (-1)^{j-1} a_{i_j} \alpha_{i_1 \cdots \hat{\imath}_j \cdots s \cdots i_m} + \sum_{j=t+1}^{m} (-1)^j a_{i_j} \alpha_{i_1 \cdots s \cdots \hat{\imath}_j \cdots i_m} \Big) \equiv 0 \pmod{(f_s, f_{i_1}, \ldots, f_{i_m})}.$$

Since ω is a k-form with $k \leq n-1$, we may assume $m \leq n-2$. The smooth complete intersection hypothesis implies that for $j \notin \{s, i_1, \ldots, i_m\}$, $f_j, f_s, f_{i_1}, \ldots, f_{i_m}$ form a regular sequence. Hence (5.18) implies

$$(5.19)\quad df^{(\delta')} \wedge df_{i_1} \wedge \cdots \wedge df_s \wedge \cdots \wedge df_{i_m} \wedge \Big((-1)^t a_s \alpha_{i_1 \cdots i_m} + \sum_{j=1}^{t} (-1)^{j-1} a_{i_j} \alpha_{i_1 \cdots \hat{\imath}_j \cdots s \cdots i_m} + \sum_{j=t+1}^{m} (-1)^j a_{i_j} \alpha_{i_1 \cdots s \cdots \hat{\imath}_j \cdots i_m} \Big) \equiv 0 \pmod{(f_s, f_{i_1}, \ldots, f_{i_m})}.$$

It now follows by Lemma 5.9 (since $b_s < a_s$ implies $a_s > 1$) that there exist homogeneous forms $\gamma^{(j)}_{i_1 \cdots i_m}$, $\delta^{(j)}_{i_1 \cdots i_m}$ such that

$$(5.20)\quad \alpha_{i_1 \cdots i_m} = \frac{(-1)^t}{a_s} \Big(\sum_{j=1}^{t} (-1)^j a_{i_j} \alpha_{i_1 \cdots \hat{\imath}_j \cdots s \cdots i_m} + \sum_{j=t+1}^{m} (-1)^{j-1} a_{i_j} \alpha_{i_1 \cdots s \cdots \hat{\imath}_j \cdots i_m} \Big) + df^{(\delta')} \wedge \gamma^{(0)}_{i_1 \cdots i_m} + df_s \wedge \gamma^{(s)}_{i_1 \cdots i_m} + f_s \delta^{(s)}_{i_1 \cdots i_m} + \sum_{j=1}^{m} df_{i_j} \wedge \gamma^{(i_j)}_{i_1 \cdots i_m} + \sum_{j=1}^{m} f_{i_j} \delta^{(i_j)}_{i_1 \cdots i_m}.$$

Such a formula holds for every m-tuple $i_1, \ldots, i_m$ not containing s. Substitute these expressions into

$$(5.21)\qquad \Theta \wedge \Big(\sum_{1 \leq i_1 < \cdots < i_m \leq r} \Omega_{i_1 \cdots i_m} \wedge \alpha_{i_1 \cdots i_m} \Big)$$

and expand. After this substitution, only alphas indexed by m-tuples containing s remain. We leave it to the reader to check that for such an m-tuple, say,

$$1 \leq j_1 < \cdots < j_t < s < j_{t+1} < \cdots < j_{m-1} \leq r,$$

the contribution to (5.21) is

$$\Theta \wedge \frac{(-1)^t \Theta}{a_s f_1 \cdots f_r} \wedge \Omega_{j_1 \cdots j_{m-1}} \wedge \alpha_{j_1 \cdots s \cdots j_{m-1}} = 0.$$

Thus after substitution from (5.20), expression (5.21) simplifies to

$$\tag{5.22} \Theta \wedge \Bigg(\sum_{\substack{1 \leq i_1 < \cdots < i_m \leq r \\ s \notin \{i_1, \ldots, i_m\}}} \Omega_{i_1 \cdots i_m} \wedge \Bigg(df^{(\delta')} \wedge \gamma^{(0)}_{i_1 \cdots i_m} + df_s \wedge \gamma^{(s)}_{i_1 \cdots i_m} + f_s \delta^{(s)}_{i_1 \cdots i_m} + \sum_{j=1}^{m} f_{i_j} \delta^{(i_j)}_{i_1 \cdots i_m} \Bigg) \Bigg).$$

We thus take

$$\tag{5.23} \tilde{\eta} = (-1)^{m+1} \Theta \wedge \Bigg(\sum_{\substack{1 \leq i_1 < \cdots < i_m \leq r \\ s \notin \{i_1, \ldots, i_m\}}} \Omega_{i_1 \cdots i_m} \wedge \gamma^{(0)}_{i_1 \cdots i_m} \Bigg) \in Z^{k-1},$$

which (by (5.14) and (5.22)) gives

$$\tag{5.24} \tilde{\omega} - df^{(\delta')} \wedge \tilde{\eta} = \Theta \wedge \Bigg(\sum_{\substack{l=0 \\ l \neq m}}^{k-1} \sum_{1 \leq i_1 < \cdots < i_l \leq r} \Omega_{i_1 \cdots i_l} \wedge \alpha_{i_1 \cdots i_l} + \sum_{\substack{1 \leq i_1 < \cdots < i_m \leq r \\ s \notin \{i_1, \ldots, i_m\}}} \Omega_{i_1 \cdots i_m} \wedge \Bigg(df_s \wedge \gamma^{(s)}_{i_1 \cdots i_m} + f_s \delta^{(s)}_{i_1 \cdots i_m} + \sum_{j=1}^{m} f_{i_j} \delta^{(i_j)}_{i_1 \cdots i_m} \Bigg) \Bigg).$$

We rewrite this as

$$\tag{5.25} \begin{aligned} \tilde{\omega} - df^{(\delta')} \wedge \tilde{\eta} = \Theta \wedge \Bigg(& \sum_{\substack{l=0 \\ l \neq m}}^{k-1} \sum_{1 \leq i_1 < \cdots < i_l \leq r} \Omega_{i_1 \cdots i_l} \wedge \alpha_{i_1 \cdots i_l} \\ & + \sum_{\substack{1 \leq i_1 < \cdots < i_m \leq r \\ s \notin \{i_1, \ldots, i_m\}}} \Omega_{i_1 \cdots i_m} \wedge f_s \delta^{(s)}_{i_1 \cdots i_m} + \sum_{\substack{1 \leq i_1 < \cdots < i_m \leq r \\ s \notin \{i_1, \ldots, i_m\}}} \Omega_{i_1 \cdots s \cdots i_m} \wedge (-1)^{m-t} f_s \gamma^{(s)}_{i_1 \cdots i_m} \\ & + \sum_{\substack{1 \leq i_1 < \cdots < i_m \leq r \\ s \notin \{i_1, \ldots, i_m\}}} \sum_{j=1}^{m} \Omega_{i_1 \cdots \hat{i}_j \cdots i_m} \wedge (-1)^{m-j} df_{i_j} \wedge \delta^{(i_j)}_{i_1 \cdots i_m} \Bigg). \end{aligned}$$

For $l \geq m+2$, the coefficient of $\Omega_{i_1 \cdots i_l}$ on the right-hand side of (5.25) equals $\alpha_{i_1 \cdots i_l}$, which is divisible by f_s by the induction hypothesis. For $l = m+1$, the coefficient of $\Omega_{i_1 \cdots i_{m+1}}$ equals $\alpha_{i_1 \cdots i_{m+1}}$ if $s \notin \{i_1, \ldots, i_{m+1}\}$, while the coefficient of $\Omega_{i_1 \cdots s \cdots i_m}$ equals $\alpha_{i_1 \cdots s \cdots i_m} + (-1)^{m-t} f_s \gamma^{(s)}_{i_1 \cdots i_m}$. In either case, it is divisible by f_s by the induction hypothesis. For $l = m$, the coefficient of $\Omega_{i_1 \cdots i_m}$ equals 0 if $s \in \{i_1, \ldots, i_m\}$ and equals $f_s \delta^{(s)}_{i_1 \cdots i_m}$ if $s \notin \{i_1, \ldots, i_m\}$. Thus the coefficient of $\Omega_{i_1 \cdots i_l}$ is divisible by f_s for all $l \geq m$, and by induction the proof of Lemma 5.11 is complete. □

Proof of Proposition 5.1. Suppose $\omega \in Z^k$ satisfies (5.2). By Corollary 5.13 we may assume that

$$\omega = df^{(\delta)} \wedge \sum_{l=1}^{k-1} \sum_{1\le i_1<\cdots<i_l\le r} \Omega_{i_1\cdots i_l} \wedge \alpha_{i_1\cdots i_l}. \tag{5.26}$$

Note that we may start the outer sum at $l = 1$ rather than at $l = 0$. We prove by induction on s that for $0 \le s \le r$ we can find $\eta_1 \in Z^{k-1}$ and $\eta_2 \in \Omega^{k-1}_{\mathbf{F}_q[x]/\mathbf{F}_q}$ such that

$$\omega - df^{(\delta')} \wedge \eta_1 - df^{(\delta)} \wedge \eta_2 = df^{(\delta)} \wedge \sum_{l=1}^{k-1} \sum_{1\le i_1<\cdots<i_l\le r} \Omega_{i_1\cdots i_l} \wedge \alpha'_{i_1\cdots i_l} \tag{5.27}$$

with all $\alpha'_{i_1\cdots i_l}$ divisible by $f_1\cdots f_s$. When $s = r$, $\Omega_{i_1\cdots i_l} \wedge \alpha'_{i_1\cdots i_l}$ has polynomial coefficients for all $i_1, \ldots, i_l$. So equation (5.27) establishes Proposition 5.1.

The assertion for $s = 0$ is immediate from (5.26). Suppose that for some s, $1 \le s \le r$, all $\alpha_{i_1\cdots i_l}$ in (5.26) are divisible by $f_1\cdots f_{s-1}$. In this case, if $i_l < s$, then $\Omega_{i_1\cdots i_l} \wedge \alpha_{i_1\cdots i_l} \in \Omega^{k-1}_{\mathbf{F}_q[x]/\mathbf{F}_q}$. So by replacing ω by $\omega - df^{(\delta)} \wedge \Omega_{i_1\cdots i_l} \wedge \alpha_{i_1\cdots i_l}$, we may assume

$$\alpha_{i_1\cdots i_l} = 0 \qquad \text{when } i_l < s. \tag{5.28}$$

We prove we can choose η_1, η_2 so that all $\alpha'_{i_1\cdots i_l}$ are divisible by $f_1\cdots f_s$ by descending induction on l. Since $\alpha_{i_1\cdots i_l} = 0$ for $l > k-1$, we may assume that for some m, $1 \le m \le k-1$, $\alpha_{i_1\cdots i_l}$ is divisible by $f_1\cdots f_s$ for $l \ge m+1$. We show that we can choose η_1, η_2 so that $\alpha'_{i_1\cdots i_l}$ is divisible by $f_1\cdots f_s$ for $l \ge m$.

Put

$$\begin{aligned} \tilde\omega &= \omega/(f_1^{a_1-1}\cdots f_r^{a_r-1}) \\ &= \Theta \wedge \Bigg(\sum_{l=1}^{k-1} \sum_{\substack{1\le i_1<\cdots<i_l\le r \\ s\le i_l}} \Omega_{i_1\cdots i_l} \wedge \alpha_{i_1\cdots i_l} \Bigg). \end{aligned} \tag{5.29}$$

We show there exist $\tilde\eta_1 \in Z^{k-1}$ and $\tilde\eta_2 \in \Omega^{k-1}_{\mathbf{F}_q[x]/\mathbf{F}_q}$ such that

$$\tilde\omega - df^{(\delta')} \wedge \tilde\eta_1 - \Theta \wedge \tilde\eta_2 = \Theta \wedge \sum_{l=1}^{k-1} \sum_{1\le i_1<\cdots<i_l\le r} \Omega_{i_1\cdots i_l} \wedge \alpha'_{i_1\cdots i_l} \tag{5.30}$$

with all $\alpha'_{i_1\cdots i_l}$ divisible by $f_1\cdots f_s$. Equation (5.27) follows from (5.30) by multiplication by $f_1^{a_1-1}\cdots f_r^{a_r-1}$.

Equation (5.2) implies

$$df^{(\delta')} \wedge \tilde\omega \equiv 0 \pmod{(f_s, f_i)} \tag{5.31}$$

for all $i \ne s$. Fix an m-tuple $1 \le i_1 < \cdots < i_m \le r$ with $s \notin \{i_1, \ldots, i_m\}$ and $s < i_m$, say, $i_t < s < i_{t+1}$. The term containing $df_{i_1} \wedge \cdots \wedge df_s \wedge \cdots \wedge df_{i_m}$ in the expansion of the right-hand side of (5.30) is given by (5.17). Using the induction hypothesis that $\alpha_{i_1\cdots i_l}$ is divisible by $f_1\cdots f_s$ for all $l \ge m+1$ and by $f_1\cdots f_{s-1}$ for all $l \ge 1$, one sees that the term containing any other product $df_{i_1} \wedge \cdots \wedge df_{i_l}$ (for $l \ge 1$) on the right-hand side of (5.30) lies in the ideal $(f_s, \{f_{i_j}\}_{i_j>s}, \{f_{i_j}^2\}_{i_j<s})$.

Since $\{i_j \mid i_j > s\} \neq \emptyset$, equation (5.31) implies that

$$df^{(\delta')} \wedge df_{i_1} \wedge \cdots \wedge df_s \wedge \cdots \wedge df_{i_m} \wedge f_1 \cdots \hat{f}_{i_1} \cdots \hat{f}_s \cdots \hat{f}_{i_m} \cdots f_r \biggl((-1)^t a_s \alpha_{i_1 \cdots i_m} + \sum_{j=1}^{t} (-1)^{j-1} a_{i_j} \alpha_{i_1 \cdots \hat{\imath}_j \cdots s \cdots i_m} + \sum_{j=t+1}^{m} (-1)^j a_{i_j} \alpha_{i_1 \cdots s \cdots \hat{\imath}_j \cdots i_m} \biggr) \equiv 0 \pmod{(f_s, \{f_{i_j}\}_{i_j > s}, \{f_{i_j}^2\}_{i_j < s})}. \tag{5.32}$$

As noted earlier, we may assume $m \leq n-2$, which implies that $f_j, f_s, f_{i_1}, \ldots, f_{i_m}$ is a regular sequence. We then deduce from (5.32) that

$$df^{(\delta')} \wedge df_{i_1} \wedge \cdots \wedge df_s \wedge \cdots \wedge df_{i_m} \wedge \biggl((-1)^t a_s \frac{\alpha_{i_1 \cdots i_m}}{f_1 \cdots f_{s-1}} + \sum_{j=1}^{t} (-1)^{j-1} a_{i_j} \frac{\alpha_{i_1 \cdots \hat{\imath}_j \cdots s \cdots i_m}}{f_1 \cdots f_{s-1}} + \sum_{j=t+1}^{m} (-1)^j a_{i_j} \frac{\alpha_{i_1 \cdots s \cdots \hat{\imath}_j \cdots i_m}}{f_1 \cdots f_{s-1}} \biggr) \equiv 0 \pmod{(f_s, f_{i_1}, \ldots, f_{i_m})}. \tag{5.33}$$

Applying Lemma 5.9 (which is permissible since $m \geq 1$) and then multiplying by $f_1 \cdots f_{s-1}$, we see that there exist homogeneous forms $\gamma^{(j)}_{i_1 \cdots i_m}$, $\delta^{(j)}_{i_1 \cdots i_m}$, all divisible by $f_1 \cdots f_{s-1}$, such that

$$\alpha_{i_1 \cdots i_m} = \frac{(-1)^t}{a_s} \biggl(\sum_{j=1}^{t} (-1)^j a_{i_j} \alpha_{i_1 \cdots \hat{\imath}_j \cdots s \cdots i_m} + \sum_{j=t+1}^{m} (-1)^{j-1} a_{i_j} \alpha_{i_1 \cdots s \cdots \hat{\imath}_j \cdots i_m} \biggr) + df^{(\delta')} \wedge \gamma^{(0)}_{i_1 \cdots i_m} + df_s \wedge \gamma^{(s)}_{i_1 \cdots i_m} + f_s \delta^{(s)}_{i_1 \cdots i_m} + \sum_{j=1}^{m} df_{i_j} \wedge \gamma^{(i_j)}_{i_1 \cdots i_m} + \sum_{j=1}^{m} f_{i_j} \delta^{(i_j)}_{i_1 \cdots i_m}. \tag{5.34}$$

Such a formula holds for every m-tuple $i_1, \ldots, i_m$ not containing s with $s < i_m$. Substitute these expressions into

$$\Theta \wedge \biggl(\sum_{\substack{1 \leq i_1 < \cdots < i_m \leq r \\ s \leq i_m}} \Omega_{i_1 \cdots i_m} \wedge \alpha_{i_1 \cdots i_m} \biggr). \tag{5.35}$$

After this substitution, only alphas indexed by m-tuples containing s remain. Consider such an m-tuple, say,

$$1 \leq j_1 < \cdots < j_t < s < j_{t+1} < \cdots < j_{m-1} \leq r.$$

If $t < m-1$, the reader may check that the contribution to (5.35) is

$$\Theta \wedge \frac{(-1)^t \Theta}{a_s f_1 \cdots f_r} \wedge \Omega_{j_1 \cdots j_{m-1}} \wedge \alpha_{j_1 \cdots s \cdots j_{m-1}} = 0.$$

In the case $t = m-1$ (i.e., $j_i < s$ for all i), the reader may check that the contribution is

$$\Theta \wedge \frac{(-1)^{m-1}}{a_s} \sum_{i=s}^{r} a_i \Omega_{ij_1 \cdots j_{m-1}} \wedge \alpha_{j_1 \cdots j_{m-1} s} = \Theta \wedge \frac{(-1)^m}{a_s} \sum_{i=1}^{s-1} a_i \Omega_{ij_1 \cdots j_{m-1}} \wedge \alpha_{j_1 \cdots j_{m-1} s},$$

since $\sum_{i=1}^r a_i \Omega_{ij_1 \cdots j_{m-1}} = \Theta \wedge \Omega_{j_1 \cdots j_{m-1}}$. Put

$$\tilde{\eta}_2 = \frac{(-1)^m}{a_s} \sum_{i=1}^{s-1} a_i \Omega_{ij_1 \cdots j_{m-1}} \wedge \alpha_{j_1 \cdots j_{m-1} s}.$$

For $i = 1, \ldots, s-1$, we have $\{i, j_1, \ldots, j_{m-1}\} \subseteq \{1, \ldots, s-1\}$, and since $\alpha_{j_1 \cdots j_{m-1} s}$ is divisible by $f_1 \cdots f_{s-1}$, we conclude that $\tilde{\eta}_2 \in \Omega^{k-1}_{\mathbf{F}_q[x]/\mathbf{F}_q}$.

Thus after substitution from (5.34), expression (5.35) simplifies to

$$\Theta \wedge \Bigg(\tilde{\eta}_2 + \sum_{\substack{1 \le i_1 < \cdots < i_m \le r \\ s \notin \{i_1, \ldots, i_m\} \\ s < i_m}} \Omega_{i_1 \cdots i_m} \wedge \Bigg(df^{(\delta')} \wedge \gamma^{(0)}_{i_1 \cdots i_m} + df_s \wedge \gamma^{(s)}_{i_1 \cdots i_m} + f_s \delta^{(s)}_{i_1 \cdots i_m} + \sum_{j=1}^m f_{i_j} \delta^{(i_j)}_{i_1 \cdots i_m} \Bigg) \Bigg). \tag{5.36}$$

We now take

$$\tilde{\eta}_1 = (-1)^{m+1} \Theta \wedge \Bigg(\sum_{\substack{1 \le i_1 < \cdots < i_m \le r \\ s \notin \{i_1, \ldots, i_m\} \\ s < i_m}} \Omega_{i_1 \cdots i_m} \wedge \gamma^{(0)}_{i_1 \cdots i_m} \Bigg) \in Z^{k-1} \tag{5.37}$$

and conclude (see (5.29)) that

$$\tilde{\omega} - df^{(\delta')} \wedge \tilde{\eta}_1 - \Theta \wedge \tilde{\eta}_2 = \Theta \wedge \Bigg(\sum_{\substack{l=0 \\ l \ne m}}^{k-1} \sum_{\substack{1 \le i_1 < \cdots < i_l \le r \\ s \le i_l}} \Omega_{i_1 \cdots i_l} \wedge \alpha_{i_1 \cdots i_l} + \sum_{\substack{1 \le i_1 < \cdots < i_m \le r \\ s \notin \{i_1, \ldots, i_m\} \\ s < i_m}} \Omega_{i_1 \cdots i_m} \wedge \Bigg(df_s \wedge \gamma^{(s)}_{i_1 \cdots i_m} + f_s \delta^{(s)}_{i_1 \cdots i_m} + \sum_{j=1}^m f_{i_j} \delta^{(i_j)}_{i_1 \cdots i_m} \Bigg) \Bigg). \tag{5.38}$$

Rewriting the right-hand side of (5.38) as in (5.25) and arguing as we did in the proof of Lemma 5.11 shows that the coefficient of $\Omega_{i_1 \cdots i_l}$ is divisible by $f_1 \cdots f_s$ for $l \ge m$. This completes the proof of Proposition 5.1. □

References

[1] A. Adolphson and S. Sperber, Exponential sums and Newton polyhedra: Cohomology and estimates, Ann. of Math. **130** (1989), 367–406. MR **91e:**11094

[2] ———, Exponential sums on $\mathbf{A}^n$, Israel J. Math. **120** (2000), 3–21. MR **2002g:**11123

[3] P. Deligne, La conjecture de Weil, I, Inst. Hautes Etudes Sci. Publ. Math. **43** (1974), 273–307. MR **49:**5013

[4] ———, La conjecture de Weil, II, Inst. Hautes Etudes Sci. Publ. Math. **52** (1980), 137–252. MR **83c:**14017

[5] J. Denef and F. Loeser, Weights of exponential sums, intersection cohomology, and Newton polyhedra, Invent. Math. **106** (1991), 275–294. MR **93a:**14019
[6] B. Dwork, On the zeta function of a hypersurface, Inst. Hautes Etudes Sci. Publ. Math. **12** (1962), 5–68. MR **28:**3039
[7] R. García López, Exponential sums and singular hypersurfaces, Manuscripta Math. **97** (1998), 45–58. MR **99e:**11112
[8] N. Katz, Sommes exponentielles, Astérisque **79** (1980), 1–209. MR **82m:**10059
[9] M. Kita, On vanishing of the twisted rational de Rham cohomology associated with hypergeometric functions, Nagoya Math. J. **135** (1994), 55–85. MR **95j:**33043
[10] H. Matsumura, Commutative Ring Theory, Cambridge University Press, Cambridge, 1986. MR **88h:**13001
[11] P. Robba, Une introduction naïve aux cohomologies de Dwork, Mémoire Soc. Math. France (nouvelle série) no. 23, 61–105, supplement to Bull. Soc. Math. France **114** (1986). MR **88a:**14023
[12] K. Saito, On a generalization of de-Rham lemma, Ann. Inst. Fourier, Grenoble **26** (1976), 165–170. MR **54:**1276
[13] J.-P. Serre, Endomorphismes complètement continus des espaces de Banach p-adiques, Inst. Hautes Etudes Sci. Publ. Math. **12** (1962), 69–85. MR **26:**1733

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